JOURNAL OF GUIDANCE, CONTROL, AND DYNAMICS
Vol. 23, No. 4, July–August 2000

Minimum-Time Constant Acceleration Orbit Transfer with First-Order Oblateness Effect

Jean Albert Kechichian*
The Aerospace Corporation, El Segundo, California 90245-4691

The analysis of the minimum-time thrust and J_2-perturbed orbit transfer between elliptic orbits of arbitrary size, shape, and orientation, and using continuous constant acceleration whose direction is optimized, is presented for two different formulations using nonsingular equinoctial orbit elements. A previous formulation that used the equinoctial orbital rotating frame for the component resolution of the perturbation vector is extended to a recently developed formulation that uses the true longitude as the sixth state variable and the polar frame for the component resolution of the thrust and J_2-induced accelerations. This new analysis is much simpler because it provides the simplest form of the differential equations for the adjoints and removes the need for solving Kepler's transcendental equation during the numerical integration of the dynamic and adjoint system of equations. Because of these simpler features, the corresponding software is also more robust resulting in improved convergence to the optimal solution of interest. The constant power and constant I_{sp} case that results in constant thrust is a trivial extension of the constant acceleration case studied here because the acceleration is easily updated by updating the mass of the vehicle during the numerical integration.

Nomenclature

a	= semimajor axis
F	= eccentric longitude, $E + \omega + \Omega$
$\boldsymbol{f}$	= acceleration vector due to J_2
$\hat{\boldsymbol{f}}, \hat{\boldsymbol{g}}, \hat{\boldsymbol{w}}$	= unit vectors along axes of direct equinoctial frame
f_t	= thrust acceleration magnitude
G	= $(1 - h^2 - k^2)^{1/2}$
h	= $e \sin(\omega + \Omega)$
J_2	= Earth second zonal harmonic, 1.08263×10^{-3}
K	= $1 + p^2 + q^2$
k	= $e \cos(\omega + \Omega)$
L	= true longitude, $\theta^* + \omega + \Omega$
n	= orbit mean motion, $\mu^{1/2} a^{-3/2}$, rad/s
p	= $\tan(i/2) \sin \Omega$
p'	= semilatus rectum, $a(1 - e^2)$
q	= $\tan(i/2) \cos \Omega$
R	= equatorial radius of the Earth, 6378.14 km
r	= radial distance
$\boldsymbol{r}$	= velocity vector
$\hat{\boldsymbol{r}}, \hat{\boldsymbol{\theta}}, \hat{\boldsymbol{h}}$	= unit vectors along axes of Euler–Hill frame
s_F, c_F	= $\sin F, \cos F$, etc.
$\boldsymbol{u}$	= unit vector in the direction of thrust
θ^*	= true anomaly
λ	= mean longitude, $M + \omega + \Omega$
μ	= Earth gravity constant, km^3/s^2

Introduction

THE problem of minimum-time orbit transfer using continuous constant acceleration whose direction is optimized, and the full set of the nonsingular equinoctial orbit elements as coordinates, has been presented previously.[1] The differential equations for the coordinates a, h, k, p, q, and λ, which were precision integrated, considered only the thrust perturbation that was resolved along the rotating direct equinoctial orbital frame, and they were generated and written in terms of F, which had to be determined through iteration at each integration step from Kepler's equation. The analysis was later made simpler by adopting the Euler–Hill or polar frame as the orbital frame for the thrust perturbation vector resolution, and the differential equations for a, h, k, p, q, and λ were written directly in terms of L, which resulted in a simpler form for the equations of the adjoint variables.[2] A further simplification was achieved[3] by adopting the Euler–Hill frame,[2] but also by replacing the sixth element λ by L and by expressing the dynamic equations in terms of L. This change of variable has provided the simplest form of the differential equations for the adjoint variables and eliminated the need to solve for F through iteration from Kepler's equation at each integration step. This optimization analysis was later extended to include the effect of the oblateness of the Earth and the J_2-induced acceleration was added to the thrust acceleration within the original mathematical framework.[1] The analysis of this augmented problem was presented[4] with the J_2 acceleration first resolved along the directions of the equinoctial frame before generating the corresponding equations for the adjoints. This paper extends the analysis of this problem by casting the mathematics within the new framework[2,3] for further verification of the results, for simplifying the equations for the adjoints, for again eliminating the need for solving Kepler's equation, and in general for producing a more robust software with better convergence characteristics. The true longitude has also been adopted as the fast element by Geffroy and Epenoy.[5] The next section shows how the full set of the dynamic and adjoint differential equations for the a, h, k, p, q, λ, and a, h, k, p, q, L sets are derived extending the original analysis[2,3] for the thrust-only case and thereby providing two additional formulations for comparison with the previous results.[4] After establishing the exact correspondence between the three various versions and duplicating a previous transfer example,[4] it is concluded that the a, h, k, p, q, L formulation, besides being the simpler one, also exhibits the more robust convergence because the transfer is generated by a single run as opposed to several runs needed with the previous software,[4] where the iterated values of the Lagrange multipliers from one run are used as the initial guess to start the next run until final convergence is achieved. Finally, for the purpose of generating averaged transfers, the averaged rate of L, due to J_2, is derived for the formulation using the a, h, k, p, q, L set, after which the analytic partial derivatives of that particular $\tilde{\dot{L}}_{J_2}$ rate, with respect to the first five slowly varying elements, are also derived.

Analysis of the Second Zonal Perturbation Effect in Minimum-Time Low-Thrust Transfers

The perturbing acceleration $\boldsymbol{f}$ due to the second zonal harmonic J_2 is resolved along the Euler–Hill axes $\hat{\boldsymbol{r}}$, $\hat{\boldsymbol{\theta}}$, and $\hat{\boldsymbol{h}}$ and given by

$$f_r = -\left(3\mu J_2 R^2 / 2r^4\right)\left(1 - 3s_i^2 s_\theta^2\right) \tag{1}$$

Received 12 April 1999; revision received 20 October 1999; accepted for publication 14 December 1999.

*Engineering Specialist, MS M4/947, P.O. Box 92957, Astrodynamics Department, Los Angeles, CA 90009; Jean.A.Kechichian@aero.org. Associate Fellow AIAA.

$$f_\theta = -\left(3\mu J_2 R^2/r^4\right) s_i^2 s_\theta c_\theta \tag{2}$$

$$f_h = -\left(3\mu J_2 R^2/r^4\right) s_i c_i s_\theta \tag{3}$$

These acceleration components given in terms of the classical elements can also be written in terms of the equinoctial elements a, h, k, p, q, and L as

$$f_r = -\frac{3\mu J_2 R^2}{2r^4}\left(1 - 12\frac{(qs_L - pc_L)^2}{(1+p^2+q^2)^2}\right) \tag{4}$$

$$f_\theta = -\frac{12\mu J_2 R^2}{r^4}\frac{(qs_L - pc_L)(qc_L + ps_L)}{(1+p^2+q^2)^2} \tag{5}$$

$$f_h = -\frac{6\mu J_2 R^2}{r^4}\frac{(qs_L - pc_L)(1-p^2-q^2)}{(1+p^2+q^2)^2} \tag{6}$$

where the radial distance is given by

$$r = \frac{a(1-h^2-k^2)}{(1+hs_L+kc_L)} \tag{7}$$

The mean longitude λ was selected as the sixth state variable[4] such that the state vector was given by $z = (ahkpq\lambda)^T$ and the equations of motion by

$$\dot a = \left(\frac{\partial a}{\partial \dot{\boldsymbol r}}\right)(\hat{\boldsymbol u} f_t + \boldsymbol f) \tag{8}$$

$$\dot h = \left(\frac{\partial h}{\partial \dot{\boldsymbol r}}\right)(\hat{\boldsymbol u} f_t + \boldsymbol f) \tag{9}$$

$$\dot k = \left(\frac{\partial k}{\partial \dot{\boldsymbol r}}\right)(\hat{\boldsymbol u} f_t + \boldsymbol f) \tag{10}$$

$$\dot p = \left(\frac{\partial p}{\partial \dot{\boldsymbol r}}\right)(\hat{\boldsymbol u} f_t + \boldsymbol f) \tag{11}$$

$$\dot q = \left(\frac{\partial q}{\partial \dot{\boldsymbol r}}\right)(\hat{\boldsymbol u} f_t + \boldsymbol f) \tag{12}$$

$$\dot \lambda = n + \left(\frac{\partial \lambda}{\partial \dot{\boldsymbol r}}\right)(\hat{\boldsymbol u} f_t + \boldsymbol f) \tag{13}$$

The partial derivatives $\partial a/\partial\dot{\boldsymbol r}$, $\partial h/\partial\dot{\boldsymbol r}$, $\partial k/\partial\dot{\boldsymbol r}$, $\partial p/\partial\dot{\boldsymbol r}$, $\partial q/\partial\dot{\boldsymbol r}$, and $\partial\lambda/\partial\dot{\boldsymbol r}$ are shown in Eqs. (7–12) in Ref. 1 and in Eqs. (41–46) in Ref. 4. They are given in terms of the orbit elements, as well as X_1, Y_1, $\dot X_1$, and $\dot Y_1$ and the partial derivatives $\partial X_1/\partial h$, $\partial X_1/\partial k$, $\partial Y_1/\partial h$, and $\partial Y_1/\partial k$. The expressions of the last six quantities are shown in Eqs. (20–23) and (47–50) in Ref. 4, and they are also listed in Ref. 1 on p. 804. X_1, Y_1, $\dot X_1$, and $\dot Y_1$ are the components of the position and velocity vectors $\boldsymbol r = X_1\hat{\boldsymbol f} + Y_1\hat{\boldsymbol g}$, and $\dot{\boldsymbol r} = \dot X_1\hat{\boldsymbol f} + \dot Y_1\hat{\boldsymbol g}$, respectively, and they are given in terms of the elements a, h, and k, as well as the eccentric longitude F. The unit vectors $\hat{\boldsymbol f}$ and $\hat{\boldsymbol g}$ are complemented by $\hat{\boldsymbol w}$ to form the direct equinoctial frame that is obtained by rotating the polar or Euler–Hill frame $(\hat{\boldsymbol r}, \hat{\boldsymbol \theta}, \hat{\boldsymbol h})$ through the angle L

$$\begin{pmatrix}\hat{\boldsymbol f}\\ \hat{\boldsymbol g}\\ \hat{\boldsymbol w}\end{pmatrix} = \begin{pmatrix} c_L & -s_L & 0\\ s_L & c_L & 0\\ 0 & 0 & 1\end{pmatrix}\begin{pmatrix}\hat{\boldsymbol r}\\ \hat{\boldsymbol \theta}\\ \hat{\boldsymbol h}\end{pmatrix} \tag{14}$$

Because the position vector $\boldsymbol r$ can also be written as $\boldsymbol r = rc_L\hat{\boldsymbol f} + rs_L\hat{\boldsymbol g}$, we have $\hat{\boldsymbol r} = (X_1/r)\hat{\boldsymbol f} + (Y_1/r)\hat{\boldsymbol g}$, $\hat{\boldsymbol\theta} = -(Y_1/r)\hat{\boldsymbol f} + (X_1/r)\hat{\boldsymbol g}$, and $\hat{\boldsymbol h} = \hat{\boldsymbol w}$, where $r = a(1 - kc_F - hs_F)$ is given in terms of F. Because of the following equivalencies, $(qs_L - pc_L) = 1/r(qY_1 - pX_1)$ and $(qc_L + ps_L) = 1/r(qX_1 + pY_1)$, the components of $\boldsymbol f$ in Eqs. (4–6) can be written directly[4] in terms of quantities that are expressed in terms of F

$$f_r = -\tfrac{3}{2}\mu J_2 R^2 r^{-4} + 18\mu J_2 R^2 r^{-6}(1+p^2+q^2)^{-2}(qY_1 - pX_1)^2 \tag{15}$$

$$f_\theta = -12\mu J_2 R^2 r^{-6}(1+p^2+q^2)^{-2}(qY_1 - pX_1)(qX_1 + pY_1) \tag{16}$$

$$f_h = -6\mu J_2 R^2 r^{-5}(1+p^2+q^2)^{-2}(1-p^2-q^2)(qY_1 - pX_1) \tag{17}$$

Thus, the $\hat{\boldsymbol f}$, $\hat{\boldsymbol g}$, and $\hat{\boldsymbol w}$ components of $\boldsymbol f$ are readily obtained from

$$f_f = (X_1/r)f_r - (Y_1/r)f_\theta \tag{18}$$

$$f_g = (Y_1/r)f_r + (X_1/r)f_\theta \tag{19}$$

$$f_w = f_h \tag{20}$$

with f_r, f_θ, and f_h given by Eqs. (15–17). The equations of motion, Eqs. (8–13) are now determined with $\boldsymbol f = f_f\hat{\boldsymbol f} + f_g\hat{\boldsymbol g} + f_w\hat{\boldsymbol w}$ resolved along the axes of the equinoctial frame, as in the case for the thrust perturbation.[4] The Hamiltonian of this differential system is given[4] by

$$H = \boldsymbol\lambda_z^T M(z, F) f_t\hat{\boldsymbol u} + \lambda_\lambda n + \boldsymbol\lambda_z^T M(z, F)\boldsymbol f \tag{21}$$

where $\boldsymbol\lambda_z^T = (\lambda_a\ \lambda_h\ \lambda_k\ \lambda_p\ \lambda_q\ \lambda_\lambda)$ is the adjoint variables vector that obeys the following Euler–Lagrange vector differential equation[4]:

$$\dot{\boldsymbol\lambda}_z = -\frac{\partial H}{\partial z} = -\boldsymbol\lambda_z^T\frac{\partial M}{\partial z}f_t\hat{\boldsymbol u} - \lambda_\lambda\frac{\partial n}{\partial z} - \boldsymbol\lambda_z^T\frac{\partial M}{\partial z}\boldsymbol f - \boldsymbol\lambda_z^T M\frac{\partial \boldsymbol f}{\partial z} \tag{22}$$

Unlike the mean longitude λ, the adjoint variables appear with an element subscript because both use the same Greek letter in the classical celestial mechanics and optimal control literatures, respectively. The differential Eqs. (8–13) are given in compact form by

$$\begin{pmatrix}\dot a\\ \dot h\\ \dot k\\ \dot p\\ \dot q\\ \dot\lambda\end{pmatrix} = \begin{pmatrix} M_{11} & M_{12} & M_{13}\\ M_{21} & M_{22} & M_{23}\\ M_{31} & M_{32} & M_{33}\\ M_{41} & M_{42} & M_{43}\\ M_{51} & M_{52} & M_{53}\\ M_{61} & M_{62} & M_{63}\end{pmatrix}\left[\begin{pmatrix}u_f\\ u_g\\ u_w\end{pmatrix} f_t + \begin{pmatrix}f_f\\ f_g\\ f_w\end{pmatrix}\right] + \begin{pmatrix}0\\0\\0\\0\\0\\n\end{pmatrix} \tag{23}$$

The $\partial M/\partial z$ partials are shown in the Appendix of Ref. 1, except that Eq. (A96) has an error in the form of an h instead of k and should read as

$$\frac{\partial^2 X_1}{\partial F\partial k} = -a\left(-(hs_F + kc_F)\frac{hk\beta^3}{1-\beta} + \frac{a^2}{r^2}(s_F - h\beta)(c_F - k) + \frac{a}{r}s_F c_F\right) \tag{24}$$

The $\partial M/\partial z$ partials in Eq. (22) was derived by using the assumption that $F = f(h, k, \lambda)$ such that

$$\frac{\partial F}{\partial a} = 0, \qquad \frac{\partial F}{\partial h} = -\frac{a}{r}c_F, \qquad \frac{\partial F}{\partial k} = \frac{a}{r}s_F, \qquad \frac{\partial F}{\partial\lambda} = \frac{a}{r} \tag{25}$$

Because r is also a function of a, h, k, and F, its partials are derived using the same assumption as earlier such that

$$\frac{\partial r}{\partial a} = \frac{a}{r}, \qquad \frac{\partial r}{\partial h} = \frac{a^2}{r}(h - s_F), \qquad \frac{\partial r}{\partial k} = \frac{a^2}{r}(k - c_F)$$

$$\frac{\partial r}{\partial\lambda} = \frac{\partial r}{\partial F}\frac{\partial F}{\partial\lambda} = \frac{a^2}{r}(ks_F - hc_F) \tag{26}$$

because

$$\frac{\partial r}{\partial F} = a(ks_F - hc_F), \qquad \frac{\partial F}{\partial\lambda} = \frac{a}{r} \tag{27}$$

The $\partial\boldsymbol f/\partial z$ partials needed in Eq. (22) involve the partial derivatives of f_r, f_θ, and f_h with respect to all six elements a, h, k, p, q, and

λ, and they have been derived in Ref. 4, where they are given by Eqs. (53–58), (59–64) and (65–70).

The equations of motion[2] were written directly in terms of L and the perturbation acceleration resolved along the rotating Euler–Hill frame, and λ was selected as the sixth element. Thus, Eqs. (13–18) of Ref. 2 were written as

$$\dot{a} = \frac{2f_t}{n(1-h^2-k^2)^{\frac{1}{2}}}[(ks_L - hc_L)u_r + (1 + hs_L + kc_L)u_\theta] \quad (28)$$

$$\dot{h} = \frac{(1-h^2-k^2)^{\frac{1}{2}}f_t}{na(1+hs_L+kc_L)}\{-(1+hs_L+kc_L)c_L u_r + [h + (2 + hs_L + kc_L)s_L]u_\theta - k(pc_L - qs_L)u_h\} \quad (29)$$

$$\dot{k} = \frac{(1-h^2-k^2)^{\frac{1}{2}}f_t}{na(1+hs_L+kc_L)}\{(1+hs_L+kc_L)s_L u_r + [k + (2 + hs_L + kc_L)c_L]u_\theta + h(pc_L - qs_L)u_h\} \quad (30)$$

$$\dot{p} = \frac{(1-h^2-k^2)^{\frac{1}{2}}f_t}{2na(1+hs_L+kc_L)}(1+p^2+q^2)s_L u_h \quad (31)$$

$$\dot{q} = \frac{(1-h^2-k^2)^{\frac{1}{2}}f_t}{2na(1+hs_L+kc_L)}(1+p^2+q^2)c_L u_h \quad (32)$$

$$\dot{\lambda} = n - \frac{(1-h^2-k^2)^{\frac{1}{2}}f_t}{na(1+hs_L+kc_L)}\Big\{\Big[\beta(1+hs_L+kc_L)(hs_L+kc_L) + 2(1-h^2-k^2)^{\frac{1}{2}}\Big]u_r + \beta(2+hs_L+kc_L)\times(hc_L - ks_L)u_\theta + (pc_L - qs_L)u_h\Big\} \quad (33)$$

Because λ is being integrated, F must be calculated from Kepler's equation, after which L is evaluated from

$$c_L = (a/r)\big[(1-h^2\beta)c_F + hk\beta s_F - k\big] \quad (34)$$

$$s_L = (a/r)\big[hk\beta c_F + (1-k^2\beta)s_F - h\big] \quad (35)$$

as shown in Eq. (19) and (20) of Ref. 2. The equations of motion for this set of state variables $z = (ahkpq\lambda)^T$ are written in a compact form as in Eq. (26) of Ref. 2,

$$\begin{pmatrix}\dot{a}\\ \dot{h}\\ \dot{k}\\ \dot{p}\\ \dot{q}\\ \dot{\lambda}\end{pmatrix} = \begin{pmatrix}B_{11} & B_{12} & B_{13}\\ B_{21} & B_{22} & B_{23}\\ B_{31} & B_{32} & B_{33}\\ B_{41} & B_{42} & B_{43}\\ B_{51} & B_{52} & B_{53}\\ B_{61} & B_{62} & B_{63}\end{pmatrix}\begin{pmatrix}u_r\\ u_\theta\\ u_h\end{pmatrix} f_t + \begin{pmatrix}0\\0\\0\\0\\0\\n\end{pmatrix} \quad (36)$$

Its Hamiltonian is given by

$$H = \boldsymbol{\lambda}_z^T B(z, L) f_t \hat{\boldsymbol{u}} + \lambda_\lambda n \quad (37)$$

and, when the J_2 perturbation is also considered, by

$$H = \boldsymbol{\lambda}_z^T B(z, L) f_t \hat{\boldsymbol{u}} + \lambda_\lambda n + \boldsymbol{\lambda}_z^T B(z, L)\boldsymbol{f} \quad (38)$$

with the components of $\boldsymbol{f}$ given by Eqs. (4–6). The unit vector $\hat{\boldsymbol{u}}$ is now resolved along the $\hat{\boldsymbol{r}}$, $\hat{\boldsymbol{\theta}}$, and $\hat{\boldsymbol{h}}$ directions, too. The Euler–Lagrange differential equations for the elements of the $\boldsymbol{\lambda}_z$ vector, namely, $\boldsymbol{\lambda}_z^T = (\lambda_a \ \lambda_h \ \lambda_k \ \lambda_p \ \lambda_q \ \lambda_\lambda)$ are given for the thrust and J_2 perturbations as

$$\dot{\boldsymbol{\lambda}}_z = -\frac{\partial H}{\partial z} = -\boldsymbol{\lambda}_z^T \frac{\partial B}{\partial z} f_t \hat{\boldsymbol{u}} - \lambda_\lambda \frac{\partial n}{\partial z} - \boldsymbol{\lambda}_z^T \frac{\partial B}{\partial z}\boldsymbol{f} - \boldsymbol{\lambda}_z^T B \frac{\partial \boldsymbol{f}}{\partial z} \quad (39)$$

The partials $\partial B/\partial z$ have a simpler form[2] than the partials $\partial M/\partial z$ appearing in Eq. (22) and shown explicitly in Ref. 1. They were derived by allowing for the variation of L with respect to h, k, and λ, and because L is expressed in terms of F as shown in Eqs. (34) and (35), F is also allowed to vary with respect to h, k, and λ (Ref. 2). Thus, $\partial B/\partial z$ was derived using

$$\frac{\partial F}{\partial a} = 0, \qquad \frac{\partial F}{\partial h} = -\frac{a}{r}c_F, \qquad \frac{\partial F}{\partial k} = \frac{a}{r}s_F, \qquad \frac{\partial F}{\partial \lambda} = \frac{a}{r} \quad (40)$$

and also using the following partials of the radial distance:

$$\frac{\partial r}{\partial a} = \frac{r}{a}, \qquad \frac{\partial r}{\partial h} = -\frac{a\big[(1-h^2\beta)s_L - hk\beta c_L\big]}{(1-h^2-k^2)^{\frac{1}{2}}}$$

$$\frac{\partial r}{\partial k} = -\frac{a\big[(1-k^2\beta)c_L - hk\beta s_L\big]}{(1-h^2-k^2)^{\frac{1}{2}}}, \qquad \frac{\partial r}{\partial \lambda} = \frac{a(ks_L - hc_L)}{(1-h^2-k^2)^{\frac{1}{2}}} \quad (41)$$

and, finally, the partial derivatives of L, namely,

$$\frac{\partial L}{\partial a} = 0$$

$$\frac{\partial L}{\partial h} = \frac{a^2/r^2(1-h^2-k^2)^{\frac{1}{2}}\big[(1-h^2\beta)s_L - hk\beta c_L\big] - s_L - (2ah/r)}{(hc_L - ks_L)}$$

$$\frac{\partial L}{\partial k} = \frac{a^2/r^2(1-h^2-k^2)^{\frac{1}{2}}\big[(1-k^2\beta)c_L - hk\beta s_L\big] - c_L - 2(a/r)k}{(hc_L - ks_L)}$$

$$\frac{\partial L}{\partial \lambda} = \frac{a^2}{r^2}(1-h^2-k^2)^{\frac{1}{2}} \quad (42)$$

The partials in Eqs. (40–42) are fully derived[2] in great detail. We still need the partials $\partial \boldsymbol{f}/\partial z$ in Eq. (39) to complete the analysis of this particular choice of the dynamic equations. From Eqs. (4–6) the following are obtained:

$$\frac{\partial f_r}{\partial a} = 6\mu J_2 R^2 r^{-5}\frac{\partial r}{\partial a} - 72\mu J_2 R^2 r^{-5}\frac{\partial r}{\partial a}(qs_L - pc_L)^2 K^{-2} \quad (43)$$

$$\frac{\partial f_r}{\partial h} = 6\mu J_2 R^2 r^{-5}\frac{\partial r}{\partial h} - 72\mu J_2 R^2 r^{-5}\frac{\partial r}{\partial h}(qc_L - ps_L)^2 K^{-2} + 36\mu J_2 R^2 r^{-4}(qs_L - pc_L)(qc_L + ps_L)\frac{\partial L}{\partial h}K^{-2} \quad (44)$$

$$\frac{\partial f_r}{\partial k} = 6\mu J_2 R^2 r^{-5}\frac{\partial r}{\partial k} - 72\mu J_2 R^2 r^{-5}\frac{\partial r}{\partial k}(qs_L - pc_L)^2 K^{-2} + 36\mu J_2 R^2 r^{-4}(qs_L - pc_L)(qc_L + ps_L)\frac{\partial L}{\partial k}K^{-2} \quad (45)$$

$$\frac{\partial f_r}{\partial p} = -36\mu J_2 R^2 r^{-4}(qs_L - pc_L)c_L K^{-2} - 72\mu J_2 R^2 r^{-4} p(qs_L - pc_L)^2 K^{-3} \quad (46)$$

$$\frac{\partial f_r}{\partial q} = 36\mu J_2 R^2 r^{-4}(qs_L - pc_L)s_L K^{-2} - 72\mu J_2 R^2 r^{-4} q(qs_L - pc_L)^2 K^{-3} \quad (47)$$

$$\frac{\partial f_r}{\partial \lambda} = 6\mu J_2 R^2 r^{-5}\frac{\partial r}{\partial \lambda} - 72\mu J_2 R^2 r^{-5}\frac{\partial r}{\partial \lambda}(qs_L - pc_L)^2 K^{-2} + 36\mu J_2 R^2 r^{-4}(qs_L - pc_L)(qc_L + ps_L)\frac{\partial L}{\partial \lambda}K^{-2} \quad (48)$$

The f_θ partials are given by

$$\frac{\partial f_\theta}{\partial a} = 48\mu J_2 R^2 r^{-5}\frac{\partial r}{\partial a}(qs_L - pc_L)(qc_L + ps_L)K^{-2} \quad (49)$$

$$\frac{\partial f_\theta}{\partial h} = 48\mu J_2 R^2 r^{-5}\frac{\partial r}{\partial h}(qs_L - pc_L)(qc_L + ps_L)K^{-2} - 12\mu J_2 R^2 r^{-4}\left[(qc_L + ps_L)^2 - (qs_L - pc_L)^2\right]\frac{\partial L}{\partial h}K^{-2} \quad (50)$$

$$\frac{\partial f_\theta}{\partial k} = 48\mu J_2 R^2 r^{-5}\frac{\partial r}{\partial k}(qs_L - pc_L)(qc_L + ps_L)K^{-2} - 12\mu J_2 R^2 r^{-4}\left[(qc_L + ps_L)^2 - (qs_L - pc_L)^2\right]\frac{\partial L}{\partial k}K^{-2} \quad (51)$$

$$\frac{\partial f_\theta}{\partial p} = -12\mu J_2 R^2 r^{-4}\left\{[s_L(qs_L - pc_L) - c_L(qc_L + ps_L)]K^{-2} - 4p(qs_L - pc_L)(qc_L + ps_L)K^{-3}\right\} \quad (52)$$

$$\frac{\partial f_\theta}{\partial q} = -12\mu J_2 R^2 r^{-4}\left\{[s_L(qc_L + ps_L) + c_L(qs_L - pc_L)]K^{-2} - 4q(qs_L - pc_L)(qc_L + ps_L)K^{-3}\right\} \quad (53)$$

$$\frac{\partial f_\theta}{\partial \lambda} = 48\mu J_2 R^2 r^{-5}\frac{\partial r}{\partial \lambda}(qs_L - pc_L)(qc_L + ps_L)K^{-2} - 12\mu J_2 R^2 r^{-4}\left[(qc_L + ps_L)^2 - (qs_L - pc_L)^2\right]\frac{\partial L}{\partial \lambda}K^{-2} \quad (54)$$

Finally, the f_h partials are given by

$$\frac{\partial f_h}{\partial a} = 24\mu J_2 R^2 r^{-5}\frac{\partial r}{\partial a}(qs_L - pc_L)(2-K)K^{-2} \quad (55)$$

$$\frac{\partial f_h}{\partial h} = 24\mu J_2 R^2 r^{-5}\frac{\partial r}{\partial h}(qs_L - pc_L)(2-K)K^{-2} - 6\mu J_2 R^2 r^{-4}(qc_L + ps_L)\frac{\partial L}{\partial h}(2-K)K^{-2} \quad (56)$$

$$\frac{\partial f_h}{\partial k} = 24\mu J_2 R^2 r^{-5}\frac{\partial r}{\partial k}(qs_L - pc_L)(2-K)K^{-2} - 6\mu J_2 R^2 r^{-4}(qc_L + ps_L)\frac{\partial L}{\partial k}(2-K)K^{-2} \quad (57)$$

$$\frac{\partial f_h}{\partial p} = -6\mu J_2 R^2 r^{-4}\left\{[-c_L(2-K) - 2p(qs_L - pc_L)]K^{-2} - 4p(qs_L - pc_L)(2-K)K^{-3}\right\} \quad (58)$$

$$\frac{\partial f_h}{\partial q} = -6\mu J_2 R^2 r^{-4}\left\{[s_L(2-K) - 2q(qs_L - pc_L)]K^{-2} - 4q(qs_L - pc_L)(2-K)K^{-3}\right\} \quad (59)$$

$$\frac{\partial f_h}{\partial \lambda} = 24\mu J_2 R^2 r^{-5}\frac{\partial r}{\partial \lambda}(qs_L - pc_L)(2-K)K^{-2} - 6\mu J_2 R^2 r^{-4}\left((qc_L + ps_L)\frac{\partial L}{\partial \lambda}(2-K)K^{-2}\right) \quad (60)$$

The partials of r and L with respect to the elements, which appear in the partial derivatives of f_r, f_θ, and f_h shown earlier, are given by Eqs. (41) and (42). Further simplifications to this analysis are possible if we use the equations of motion developed previously[3] and that use L as the sixth state variable. The first five equations are identical to Eqs. (28–32) and the sixth equation given by Eq. (19) or (20) of Ref. 3, or

$$\dot{L} = \frac{na^2(1-h^2-k^2)^{\frac{1}{2}}}{r^2} + \frac{r(qs_L - pc_L)}{na^2(1-h^2-k^2)^{\frac{1}{2}}}f_h \quad (61)$$

The basic dynamic equations for the thrust perturbation are written in compact form as

$$\begin{pmatrix}\dot{a}\\ \dot{h}\\ \dot{k}\\ \dot{p}\\ \dot{q}\\ \dot{L}\end{pmatrix} = \begin{pmatrix} B^L_{11} & B^L_{12} & B^L_{13}\\ B^L_{21} & B^L_{22} & B^L_{23}\\ B^L_{31} & B^L_{32} & B^L_{33}\\ B^L_{41} & B^L_{42} & B^L_{43}\\ B^L_{51} & B^L_{52} & B^L_{53}\\ B^L_{61} & B^L_{62} & B^L_{63}\end{pmatrix}\begin{pmatrix}u_r\\ u_\theta\\ u_h\end{pmatrix} f_t + \begin{pmatrix}0\\0\\0\\0\\0\\ na^2(1-h^2-k^2)^{\frac{1}{2}}/r^2\end{pmatrix} \quad (62)$$

The Hamiltonian involving the thrust perturbation is given by

$$H = \boldsymbol{\lambda}_z^T B^L(z) f_t \boldsymbol{u} + \lambda_L[na^2(1-h^2-k^2)^{\frac{1}{2}}/r^2]$$

and for the combined thrust and J_2 perturbations by

$$H = \boldsymbol{\lambda}_z^T B^L(z) f_t \boldsymbol{u} + \lambda_L[na^2(1-h^2-k^2)^{\frac{1}{2}}/r^2] + \boldsymbol{\lambda}_z^T B^L(z)\boldsymbol{f} \quad (63)$$

The adjoint equations take the form

$$\dot{\boldsymbol{\lambda}}_z = -\frac{\partial H}{\partial z} = -\boldsymbol{\lambda}_z^T\frac{\partial B^L}{\partial z}f_t\hat{\boldsymbol{u}} - \lambda_L\frac{\partial}{\partial z}\left(\frac{na^2(1-h^2-k^2)^{\frac{1}{2}}}{r^2}\right) - \boldsymbol{\lambda}_z^T\frac{\partial B^L}{\partial z}\boldsymbol{f} - \boldsymbol{\lambda}_z^T B^L\frac{\partial \boldsymbol{f}}{\partial z} \quad (64)$$

As shown previously,[3] the partials of L with respect to a, h, and k are identically equal to zero, with $\partial L/\partial L = 1$, and the partials of r with respect to a, h, k, and L are given by

$$\frac{\partial r}{\partial a} = \frac{r}{a}, \qquad \frac{\partial r}{\partial h} = -\frac{r}{a(1-h^2-k^2)}(2ah + rs_L)$$

$$\frac{\partial r}{\partial k} = -\frac{r}{a(1-h^2-k^2)}(2ak + rc_L)$$

$$\frac{\partial r}{\partial L} = -\frac{r^2}{a}\frac{(hc_L - ks_L)}{(1-h^2-k^2)} \quad (65)$$

The 6×3 matrix B^L as well as its $\partial B^L/\partial z$ partials are listed in Ref. 3, and this formulation using the state vector $z = (a\ h\ k\ p\ q\ L)^T$ provides the simplest form of the $\partial B^L/\partial z$ partials when compared to either the $\partial B/\partial z$ or the $\partial M/\partial z$ partials because, unlike $M(z, F)$ and $B(z, L)$, the present matrix B^L is only a direct function of z, namely, $B^L(z)$, which allows us to consider $\partial L/\partial a = \partial L/\partial h = \partial L/\partial k = 0$, thus simplifying to a considerable extent the form taken by the differential equations for the adjoints. This simplification is still effective when the $\partial \boldsymbol{f}/\partial z$ partials are generated in Eq. (64) because there, too, we are allowed to ignore the variation of L with respect to a, h, and k. These partial derivatives are now written explicitly from Eqs. (4–6) as

$$\frac{\partial f_r}{\partial a} = 6\mu J_2 R^2 r^{-5}\frac{\partial r}{\partial a} - 72\mu J_2 R^2 r^{-5}\frac{\partial r}{\partial a}(qs_L - pc_L)^2K^{-2} \quad (66)$$

$$\frac{\partial f_r}{\partial h} = 6\mu J_2 R^2 r^{-5}\frac{\partial r}{\partial h} - 72\mu J_2 R^2 r^{-5}\frac{\partial r}{\partial h}(qs_L - pc_L)^2K^{-2} \quad (67)$$

$$\frac{\partial f_r}{\partial k} = 6\mu J_2 R^2 r^{-5}\frac{\partial r}{\partial k} - 72\mu J_2 R^2 r^{-5}\frac{\partial r}{\partial k}(qs_L - pc_L)^2K^{-2} \quad (68)$$

$$\frac{\partial f_r}{\partial p} = -36\mu J_2 R^2 r^{-4}(qs_L - pc_L)c_L K^{-2} - 72\mu J_2 R^2 r^{-4}p(qs_L - pc_L)^2K^{-3} \quad (69)$$

$$\frac{\partial f_r}{\partial q} = 36\mu J_2 R^2 r^{-4}(qs_L - pc_L)s_L K^{-2} - 72\mu J_2 R^2 r^{-4}q(qs_L - pc_L)^2K^{-3} \quad (70)$$

Table 1 Orbit parameters

Orbit	a, km	e	i, deg	Ω, deg	ω, deg	M, deg
Initial	7000	0	28.5	0	0	−131.7396776 (optimized)
Target	42000	10^{-3}	1	0	0	Free
Achieved	41999.99992	1.000022×10^{-3}	1.000001	359.999569	359.999132	45.411543 (optimized)

$$\frac{\partial f_r}{\partial L}=6\mu J_2R^2r^{-5}\frac{\partial r}{\partial L}-72\mu J_2R^2r^{-5}\frac{\partial r}{\partial L}(qs_L-pc_L)^2K^{-2}+36\mu J_2R^2r^{-4}(qs_L-pc_L)(qc_L+ps_L)K^{-2}\tag{71}$$

The f_θ partials are given by

$$\frac{\partial f_\theta}{\partial a}=48\mu J_2R^2r^{-5}\frac{\partial r}{\partial a}(qs_L-pc_L)(qc_L+ps_L)K^{-2}\tag{72}$$

$$\frac{\partial f_\theta}{\partial h}=48\mu J_2R^2r^{-5}\frac{\partial r}{\partial h}(qs_L-pc_L)(qc_L+ps_L)K^{-2}\tag{73}$$

$$\frac{\partial f_\theta}{\partial k}=48\mu J_2R^2r^{-5}\frac{\partial r}{\partial k}(qs_L-pc_L)(qc_L+ps_L)K^{-2}\tag{74}$$

$$\frac{\partial f_\theta}{\partial p}=-12\mu J_2R^2r^{-4}\big\{[s_L(qs_L-pc_L)-c_L(qc_L+ps_L)]K^{-2}-4p(qs_L-pc_L)(qc_L+ps_L)K^{-3}\big\}\tag{75}$$

$$\frac{\partial f_\theta}{\partial q}=-12\mu J_2R^2r^{-4}\big\{[s_L(qc_L+ps_L)+c_L(qs_L-pc_L)]K^{-2}-4q(qs_L-pc_L)(qc_L+ps_L)K^{-3}\big\}\tag{76}$$

$$\frac{\partial f_\theta}{\partial L}=48\mu J_2R^2r^{-5}\frac{\partial r}{\partial L}(qs_L-pc_L)(qc_L+ps_L)K^{-2}-12\mu J_2R^2r^{-4}\big[(qc_L+ps_L)^2-(qs_L-pc_L)^2\big]K^{-2}\tag{77}$$

Finally, the f_h partials are given by

$$\frac{\partial f_h}{\partial a}=24\mu J_2R^2r^{-5}\frac{\partial r}{\partial a}(qs_L-pc_L)(2-K)K^{-2}\tag{78}$$

$$\frac{\partial f_h}{\partial h}=24\mu J_2R^2r^{-5}\frac{\partial r}{\partial h}(qs_L-pc_L)(2-K)K^{-2}\tag{79}$$

$$\frac{\partial f_h}{\partial k}=24\mu J_2R^2r^{-5}\frac{\partial r}{\partial k}(qs_L-pc_L)(2-K)K^{-2}\tag{80}$$

$$\frac{\partial f_h}{\partial p}=-6\mu J_2R^2r^{-4}\big\{[-c_L(2-K)-2p(qs_L-pc_L)]K^{-2}-4p(qs_L-pc_L)(2-K)K^{-3}\big\}\tag{81}$$

$$\frac{\partial f_h}{\partial q}=-6\mu J_2R^2r^{-4}\big\{[s_L(2-K)-2q(qs_L-pc_L)]K^{-2}-4q(qs_L-pc_L)(2-K)K^{-3}\big\}\tag{82}$$

$$\frac{\partial f_h}{\partial L}=24\mu J_2R^2r^{-5}\frac{\partial r}{\partial L}(qs_L-pc_L)(2-K)K^{-2}-6\mu J_2R^2r^{-4}(qc_L+ps_L)(2-K)K^{-2}\tag{83}$$

The partial derivatives of r, namely, $\partial r/\partial a$, $\partial r/\partial h$, $\partial r/\partial k$, and $\partial r/\partial L$ that appear in Eqs. (66–83) are according to Eqs. (65). Thus, the combined use of the polar or Euler–Hill frame instead of the more cumbersome equinoctial $\hat{\boldsymbol f}$, $\hat{\boldsymbol g}$, $\hat{\boldsymbol w}$ frame, and the use of L as the sixth state variable instead of λ, has eliminated the need to calculate F from Kepler's equation by iteration at each integration step and has resulted in the simplest form of the equations for the adjoint differential equations that are integrated numerically and simultaneously with the dynamic equations. The optimal orientation of the thrust acceleration vector is provided by the optimal control $\hat{\boldsymbol u}=[\boldsymbol\lambda_z^T B^L(z)]^T/|\boldsymbol\lambda_z^T B^L(z)|$ in this case. Because of the error in the $\partial^2 X_1/\partial F\partial k$ partial, the example shown in Ref. 4 is generated to provide the comparison needed to validate the versions that have been derived earlier for the case where L effectively replaces F in the right-hand sides of the dynamic and adjoint equations. The initial and target orbit parameters are shown in Table 1.

This free–free transfer is solved by guessing the initial values of the first five Lagrange multipliers, namely, $(\lambda_a)_0$, $(\lambda_h)_0$, $(\lambda_k)_0$, $(\lambda_p)_0$, and $(\lambda_q)_0$, as well as the initial longitude $(\lambda)_0$ and the transfer time t_f along with imposed $(\lambda_\lambda)_0=0$, and by iterating on the guessed quantities until the final parameters a_f, h_f, k_f, p_f, q_f, which correspond to the target orbit shown in Table 1, and $(\lambda_\lambda)_f=0$ and $H_f=1$ are met to within a small tolerance. The orientation of the continuous constant acceleration vector is obtained from Pontryagin's maximum principle by selecting $\hat{\boldsymbol u}$ parallel to $\boldsymbol\lambda_z^T M(z,F)$. In practice an unconstrained minimization algorithm is used to minimize the quadratic objective function F'

$$F'=w_1(a-a_f)^2+w_2(h-h_f)^2+w_3(k-k_f)^2+w_4(p-p_f)^2+w_5(q-q_f)^2+w_6(\lambda_\lambda-0)^2+w_7(H-1)^2\tag{84}$$

where the w_i are adjustable weights. Let $f_t=9.8\times10^{-5}$ km/s^2, and use the solution corresponding to the minimum-time transfer without the J_2 perturbation,[2,3] namely,

$$(\lambda_a)_0=4.675229762\ \text{s/km},\qquad(\lambda_p)_0=1.778011878\times10^1\ \text{s}$$
$$(\lambda_h)_0=5.413413947\times10^2\ \text{s},\qquad(\lambda_q)_0=-2.258455855\times10^4\ \text{s}$$
$$(\lambda_k)_0=-9.202702084\times10^3\ \text{s},\qquad(\lambda)_0=-2.274742851\ \text{rad}$$
$$t_f=58089.90058\ \text{s}\tag{85}$$

with $H_f=1.003704$. The value of $(\lambda)_0$ corresponds to $M_0=-130.333164$ deg, and the multipliers in Eq. (85) are scaled to provide $H=1$ resulting in

$$(\lambda_a)_0=4.657973438\ \text{s/km},\qquad(\lambda_p)_0=1.771449217\times10^1\ \text{s}$$
$$(\lambda_h)_0=5.393432977\times10^2\ \text{s},\qquad(\lambda_q)_0=-2.250119870\times10^4\ \text{s}$$
$$(\lambda_k)_0=-9.168734810\times10^3\ \text{s},\qquad(\lambda)_0=-2.274742851\ \text{rad}$$
$$t_f=58089.90058\ \text{s}\tag{86}$$

An open-loop propagation is now carried out using Eqs. (8–13) and Eq. (22) with the optimal control law obtained from $\hat{\boldsymbol u}=\boldsymbol\lambda_z^T M(z,F)/|\boldsymbol\lambda_z^T M(z,F)|$ and starting at time zero with the initial values in Eq. (86) with $(\lambda_\lambda)_0=0$ and terminating the numerical integration at $t_f=58089.90058$ s, with $J_2=1.08263\times10^{-3}$. The initial conditions corresponding to the minimum-time solution without the J_2 effect will fail to integrate the orbit to the desired target orbit if we activate the J_2 perturbation as we just did. This failure is shown in Fig. 1, which shows the evolution of the eccentricity vs the semimajor axis with $a_f=40427.5184$ km and $e_f=4.024759\times10^{-2}$ well short of the desired parameters $a_f=42000$ km and $e_f=10^{-3}$ at time t_f.

The minimum-time solution with J_2 active is now obtained by iteration starting from the guess provided in Eq. (86) and still using the nonlinear system of equations that correspond to Eqs. (22) and (23). The result is the solution given by

$$(\lambda_a)_0=4.800100306\ \text{s/km},\qquad(\lambda_p)_0=3.281827358\times10^1\ \text{s}$$
$$(\lambda_h)_0=8.060772261\times10^2\ \text{s},\qquad(\lambda_q)_0=-2.254928992\times10^4\ \text{s}$$
$$(\lambda_k)_0=-9.150040837\times10^3\ \text{s},\qquad(\lambda)_0=-2.299291130\ \text{rad}$$
$$t_f=58104.83438\ \text{s}\tag{87}$$

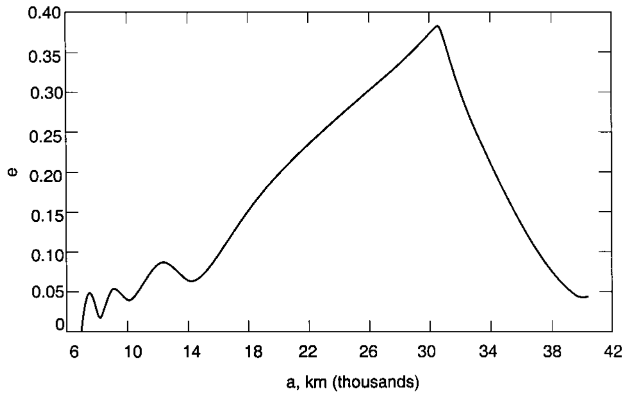

Fig. 1 Effects of J_2 perturbation on minimum-time transfer solution generated without oblateness consideration.

The value of $(\lambda)_0$ corresponds to $M_0 = -131.7396776$ deg, and the final achieved parameters are shown in Table 1 with $M_f = 45.411543$ deg with imposed $(\lambda_\lambda)_0 = 0$ s/rad and iterated $(\lambda_\lambda)_f = 4.07864 \times 10^{-5}$ s/rad, which is effectively near the desired value of zero for an optimized arrival point on the target orbit. The Hamiltonian $H = 1.000000$ is constant throughout, and the total $\Delta V = f_t \cdot t_f$ is equal to 5.694273769 km/s. This solution is slightly different from the one shown in Ref. 4 because there the error in the $\partial^2 X_1 / \partial F \partial k$ has affected and contaminated the exact solution here. The net effect in considering the J_2 perturbation has resulted in a transfer duration increased by some 14.9338 s when compared to the thrust only case, and the location of the departure point has now moved by some 1.406 deg with respect to the location corresponding to the solution without the oblateness effect. We now verify the soundness of the last two versions of this optimization problem by carrying out open-loop integrations of the system of equations corresponding to the Hamiltonian in Eq. (38), namely,

$$\begin{pmatrix}\dot a\\ \dot h\\ \dot k\\ \dot p\\ \dot q\\ \dot\lambda\end{pmatrix} = \begin{pmatrix}B_{11} & B_{12} & B_{13}\\ B_{21} & B_{22} & B_{23}\\ B_{31} & B_{32} & B_{33}\\ B_{41} & B_{42} & B_{43}\\ B_{51} & B_{52} & B_{53}\\ B_{61} & B_{62} & B_{63}\end{pmatrix}\left[\begin{pmatrix}u_r\\ u_\theta\\ u_h\end{pmatrix} f_t + \begin{pmatrix}f_r\\ f_\theta\\ f_h\end{pmatrix}\right] + \begin{pmatrix}0\\0\\0\\0\\0\\n\end{pmatrix} \tag{88}$$

and Eq. (39), and starting from the initial conditions in Eq. (87), until the final time $t_f = 58104.83438$ s. The trajectory is integrated to $a_f = 41999.99992$ km, $e_f = 1.000022 \times 10^{-3}$, $i_f = 1.000001$ deg, $\Omega_f = 359.999569$ deg, $\omega_f = 359.999137$ deg, and $M_f = 45.411538$ deg matching the achieved parameters in Table 1 very closely. In a similar way, the dynamic equations corresponding to the Hamiltonian in Eq. (63), namely,

$$\begin{pmatrix}\dot a\\ \dot h\\ \dot k\\ \dot p\\ \dot q\\ \dot L\end{pmatrix} = \begin{pmatrix}B^L_{11} & B^L_{12} & B^L_{13}\\ B^L_{21} & B^L_{22} & B^L_{23}\\ B^L_{31} & B^L_{32} & B^L_{33}\\ B^L_{41} & B^L_{42} & B^L_{43}\\ B^L_{51} & B^L_{52} & B^L_{53}\\ B^L_{61} & B^L_{62} & B^L_{63}\end{pmatrix}\left[\begin{pmatrix}u_r\\ u_\theta\\ u_h\end{pmatrix} f_t + \begin{pmatrix}f_r\\ f_\theta\\ f_h\end{pmatrix}\right] + \begin{pmatrix}0\\0\\0\\0\\0\\ na^2(1-h^2-k^2)^{\frac{1}{2}}/r^2\end{pmatrix} \tag{89}$$

as well as the adjoint system in Eq. (64) are also integrated forward in time starting with the same initial conditions listed in Eq. (87), where $(L)_0 = -2.299291130$ rad replaces the identical value of $(\lambda)_0$ shown there, with $(\lambda_L)_0 = 0$ instead of $(\lambda_\lambda)_0$, until $t_f = 58104.83438$ s. The trajectory is integrated to $a_f = 41999.99992$ km, $e_f = 1.000022 \times 10^{-3}$, $i_f = 1.000001$ deg, $\Omega_f = 359.999569$ deg, $\omega_f = 359.999137$ deg, and $M_f = 45.411538$ deg, effectively showing that all three formulations are identical and provide the same trajectory. All of the pertinent integrations are carried out by setting the relative and absolute error controls at the 10^{-9} level. The solution generated in Eq. (87) required several runs with different w_i weights for each run for a successful convergence. However, the converged solution was obtained with a single run using the L formulation developed in Eqs. (63), (89), and (64) with $w_1 = 1$, $w_2 = 10^7$, $w_3 = 10^8$, $w_4 = 10^7$, $w_5 = 10^8$, $w_6 = 1$, and $w_7 = 10^2$ and with the function value F', where the sixth term in Eq. (84) is now replaced by $w_6(\lambda_L - 1)^2$, minimized to the level of 10^{-3}. The minimization algorithm uses an internal tolerance value for the convergence tests that is equal to the square root of the error in the function values, which are essentially equal to the machine roundoff level $\varepsilon_{\text{mach}}$. Unlike the algorithms that compute the zeros of a function to nearly full machine precision, the minimization algorithms can find the local minima to only about half-precision because now the zeros of the derivatives of F' are being computed. This single run required about 85 min on a 66-MHz machine, 65 min on a 266-MHz machine, and about 35 min on a 1000-MHz double Pentium III personal computer. Another advantage of the L formulation for minimum-time problems lies in the increased compactness of the resulting equations, which would be further beneficial if other perturbations such as those due to third body gravity are also considered. Finally, it is possible to modulate the value of J_2 from zero to its actual value and obtain a series of converged runs starting from the known thrust-only solution, if the nominal J_2-perturbed case proved difficult to generate.

We finally compare this J_2-optimized trajectory given in Eq. (87) with the trajectory that is generated by using the initial conditions shown in Eq. (85) that corresponds to the minimum-time transfer solution for the thrust perturbation only. However, we integrate the same system of Eqs. (22) and (23) with the J_2 perturbation on until $t_f = 58089.90058$ s to compare the time histories of the various transfer parameters even for this short-duration transfer, where J_2 introduces only a small integrated perturbation effect as compared to the roughly 10^{-2}-g thrust acceleration used here. Figures 2 and 3 compare the evolutions of the equinoctial elements k and p, whereas Figs. 4 and 5 compare the evolution of the classical elements e and M showing clearly that the fully J_2-optimized solution achieves the required target conditions. Figures 6–8 show the evolutions of the Lagrange multipliers λ_k, λ_p, and λ_λ effectively showing that the optimal $(\lambda_\lambda)_f$ alone terminates at the zero value as it should for a free–free transfer in which both the initial and final orbital locations are indeed optimal, providing the absolute minimum-time transfer solution. In Fig. 9, only the optimal transfer satisfies the $H = 1$ value throughout the transfer, whereas the other transfer results in a slightly lower value of the Hamiltonian, indicating that the transfer time is not optimized and that in this case is optimistic.

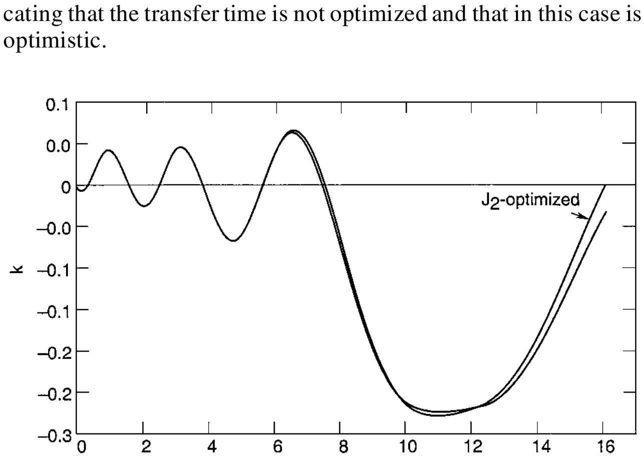

Fig. 2 Evolution of k vs time for J_2-optimized and J_2-nonoptimized perturbed minimum-time transfers.

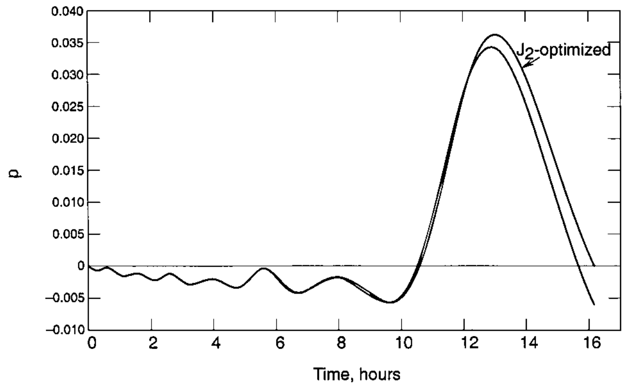

Fig. 3 Evolution of p vs time for J_2-optimized and J_2-nonoptimized perturbed minimum-time transfers.

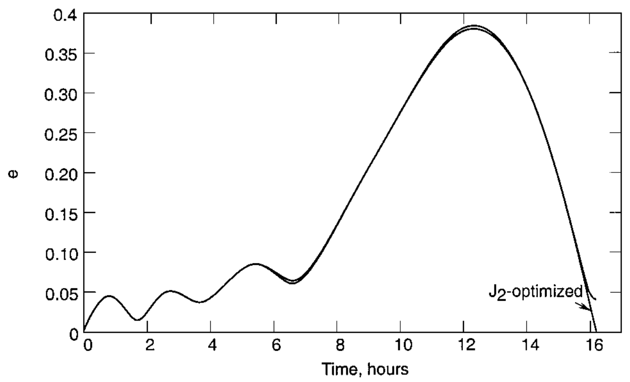

Fig. 4 Evolution of e vs time for J_2-optimized and J_2-nonoptimized perturbed minimum-time transfers.

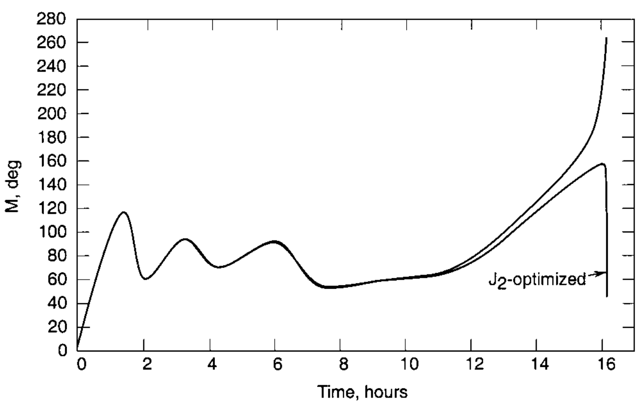

Fig. 5 Evolution of M vs time for J_2-optimized and J_2-nonoptimized perturbed minimum-time transfers.

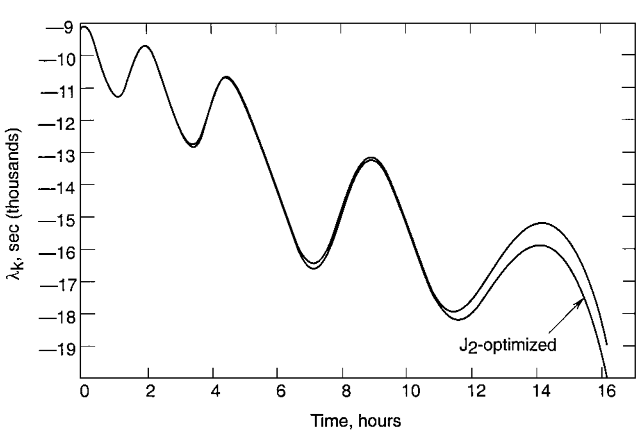

Fig. 6 Evolution of λ_k vs time for J_2-optimized and J_2-nonoptimized perturbed minimum-time transfers.

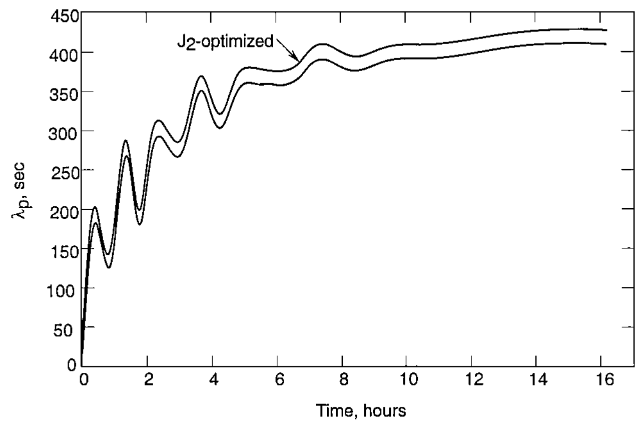

Fig. 7 Evolution of λ_p vs time for J_2-optimized and J_2-nonoptimized perturbed minimum-time transfers.

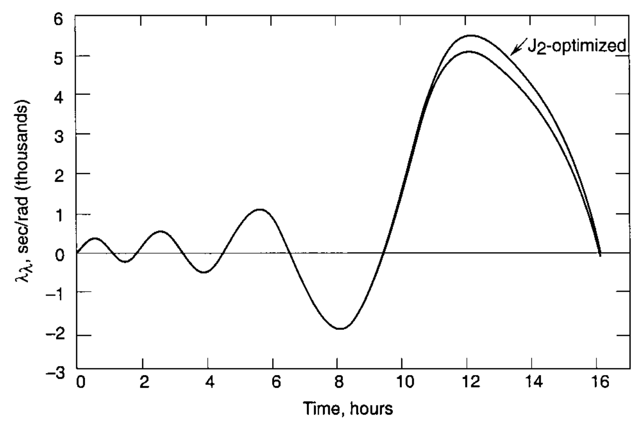

Fig. 8 Evolution of λ_λ vs time for J_2-optimized and J_2-nonoptimized perturbed minimum-time transfers.

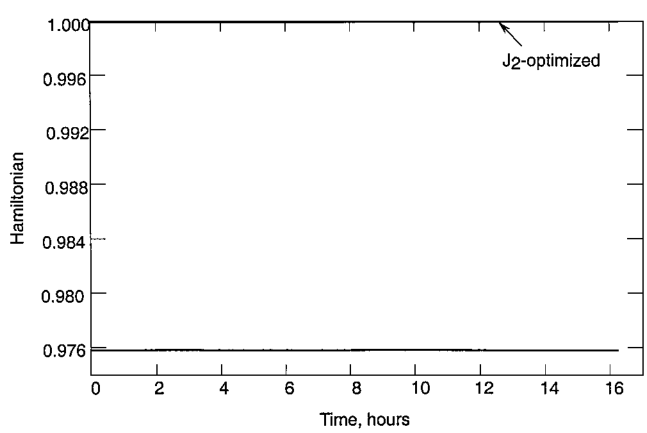

Fig. 9 Evolution of Hamiltonian vs time for J_2-optimized and J_2-nonoptimized perturbed minimum-time transfers.

Averaged Rates of the Elements Due to J_2 and Their Partial Derivatives

The averaged Hamiltonian for the thrust and J_2 perturbations can be written as

$$\tilde{H} = \tilde{H}_T + \tilde{H}_{J_2} \tag{90}$$

where

$$\tilde{H}_{J_2} = \tilde{\lambda}_z^T \dot{\tilde{z}}_{J_2} \tag{91}$$

The differential equations for the adjoints involve the $\partial \dot{\tilde{z}}_{J_2} / \partial \tilde{z}$ partial derivatives, and both $\dot{\tilde{z}}_{J_2}$ and $\partial \dot{\tilde{z}}_{J_2} / \partial \tilde{z}$ have been derived for the

$\tilde{a}, \tilde{h}, \tilde{k}, \tilde{p}, \tilde{q}$ and $\tilde{\lambda}$ elements.[4] Because we are now replacing $\tilde{\lambda}$ by $\tilde{L}$, an expression for $\dot{\tilde{L}}_{J_2}$ must be developed first before its partial derivatives can be generated. By the use of the averaged rates of ω, Ω, and M in terms of the classical elements,[4]

$$\dot{\tilde{\omega}}_{J_2} = \tfrac{3}{2} J_2 (R^2/\tilde{p}'^2)\tilde{n}\left[2 - \tfrac{5}{2}s_{\tilde{i}}^2\right] \tag{92}$$

$$\dot{\tilde{\Omega}}_{J_2} = -\tfrac{3}{2} J_2 (R^2/\tilde{p}'^2)\tilde{n}c_{\tilde{i}} \tag{93}$$

$$\dot{\tilde{M}}_{J_2} = \tfrac{3}{2} J_2 (R^2/\tilde{p}'^2)\tilde{n}\left[1 - \tfrac{3}{2}s_{\tilde{i}}^2\right](1-\tilde{e}^2)^{\frac{1}{2}} \tag{94}$$

The following expressions for $\dot{\tilde{h}}_{J_2}$, $\dot{\tilde{k}}_{J_2}$, $\dot{\tilde{p}}_{J_2}$, $\dot{\tilde{q}}_{J_2}$, and $\dot{\tilde{\lambda}}_{J_2}$ were developed by also making use of $\dot{\tilde{a}}_{J_2} = \dot{\tilde{e}}_{J_2} = \dot{\tilde{i}}_{J_2} = 0$ and by noting that $\tilde{p}'$ is the mean orbit parameter given by $\tilde{p}' = \tilde{a}(1-\tilde{e}^2)$:

$$\dot{\tilde{h}}_{J_2} = \frac{3}{2}\frac{\tilde{k}J_2R^2\mu}{\tilde{n}\tilde{a}^5(1-\tilde{h}^2-\tilde{k}^2)^2}\left(\frac{1+3(\tilde{p}^2+\tilde{q}^2)^2-6(\tilde{p}^2+\tilde{q}^2)}{(1+\tilde{p}^2+\tilde{q}^2)^2}\right) \tag{95}$$

$$\dot{\tilde{k}}_{J_2} = -\frac{3}{2}\frac{\tilde{h}J_2R^2\mu}{\tilde{n}\tilde{a}^5(1-\tilde{h}^2-\tilde{k}^2)^2}\left(\frac{1+3(\tilde{p}^2+\tilde{q}^2)^2-6(\tilde{p}^2+\tilde{q}^2)}{(1+\tilde{p}^2+\tilde{q}^2)^2}\right) \tag{96}$$

$$\dot{\tilde{p}}_{J_2} = -\frac{3}{2}\frac{\tilde{q}J_2R^2\mu(1-\tilde{p}^2-\tilde{q}^2)}{\tilde{n}\tilde{a}^5(1-\tilde{h}^2-\tilde{k}^2)^2(1+\tilde{p}^2+\tilde{q}^2)} \tag{97}$$

$$\dot{\tilde{q}}_{J_2} = \frac{3}{2}\frac{\tilde{p}J_2R^2\mu(1-\tilde{p}^2-\tilde{q}^2)}{\tilde{n}\tilde{a}^5(1-\tilde{h}^2-\tilde{k}^2)^2(1+\tilde{p}^2+\tilde{q}^2)} \tag{98}$$

$$\dot{\tilde{\lambda}}_{J_2} = \dot{\tilde{M}}_{J_2} + \left(\dot{\tilde{h}}_{J_2}\big/\tilde{k}\right) = \dot{\tilde{M}}_{J_2} - \left(\dot{\tilde{k}}_{J_2}\big/\tilde{h}\right) \tag{99}$$

where

$$\dot{\tilde{M}}_{J_2} = \frac{3}{2}\frac{J_2R^2\mu(1-\tilde{h}^2-\tilde{k}^2)^{-\frac{3}{2}}}{\tilde{n}\tilde{a}^5(1+\tilde{p}^2+\tilde{q}^2)^2}\left[1+(\tilde{p}^2+\tilde{q}^2)^2-4(\tilde{p}^2+\tilde{q}^2)\right] \tag{100}$$

There is a typographical error in Eq. (108) of Ref. 4, which should have $(1+\tilde{p}^2+\tilde{q}^2)^2$ instead of $(1+\tilde{p}^2+\tilde{q}^2)$ as the term in the denominator, and this was corrected in Eq. (96) here. Equations (95–100) are derived as Eqs. (107–110) and (112), (111), respectively in Ref. 4 where two typographical errors in Eq. (111) of Ref. 4 have been corrected in Eq. (100) here. The partial derivatives of $\dot{\tilde{h}}_{J_2}, \dot{\tilde{k}}_{J_2}, \dot{\tilde{p}}_{J_2}, \dot{\tilde{q}}_{J_2}$, and $\dot{\tilde{\lambda}}_{J_2}$ with respect to $\tilde{a}$, $\tilde{h}$, $\tilde{k}$, $\tilde{p}$, and $\tilde{q}$ have also been derived in Ref. 4 and are given there by Eqs. (113–137). Note that Eq. (135) of Ref. 4 also has a typographical error and should read

$$\frac{\partial\dot{\tilde{\lambda}}_{J_2}}{\partial\tilde{k}} = \frac{3\tilde{k}\dot{\tilde{M}}_{J_2}}{(1-\tilde{h}^2-\tilde{k}^2)} - \frac{4\tilde{k}}{(1-\tilde{h}^2-\tilde{k}^2)}\frac{\dot{\tilde{k}}_{J_2}}{\tilde{h}} \tag{101}$$

whereas Eqs. (136) and (137) should read

$$\frac{\partial\dot{\tilde{\lambda}}_{J_2}}{\partial\tilde{p}} = \frac{12J_2R^2\mu\tilde{p}[3(\tilde{p}^2+\tilde{q}^2)-2]}{\tilde{n}\tilde{a}^5(1-\tilde{h}^2-\tilde{k}^2)^2(1+\tilde{p}^2+\tilde{q}^2)^3} + \frac{6J_2R^2\mu\tilde{p}(1-\tilde{h}^2-\tilde{k}^2)^{-\frac{3}{2}}[3(\tilde{p}^2+\tilde{q}^2)-3]}{\tilde{n}\tilde{a}^5(1+\tilde{p}^2+\tilde{q}^2)^3} \tag{102}$$

$$\frac{\partial\dot{\tilde{\lambda}}_{J_2}}{\partial\tilde{q}} = \frac{12J_2R^2\mu\tilde{q}[3(\tilde{p}^2+\tilde{q}^2)-2]}{\tilde{n}\tilde{a}^5(1-\tilde{h}^2-\tilde{k}^2)^2(1+\tilde{p}^2+\tilde{q}^2)^3} + \frac{6J_2R^2\mu\tilde{q}(1-\tilde{h}^2-\tilde{k}^2)^{-\frac{3}{2}}[3(\tilde{p}^2+\tilde{q}^2)-3]}{\tilde{n}\tilde{a}^5(1+\tilde{p}^2+\tilde{q}^2)^3} \tag{103}$$

It can also be shown that $\dot{\tilde{M}}_{0_{J_2}} = \dot{\tilde{M}}_{J_2}$ and $\dot{\tilde{\lambda}}_{0_{J_2}} = \dot{\tilde{\lambda}}_{J_2}$. From $\dot{M}_0$ written in terms of the partial derivatives of the disturbing function R, namely,

$$\dot{M}_0 = -\frac{2}{na}\frac{\partial R}{\partial a} - \frac{(1-e^2)}{na^2e}\frac{\partial R}{\partial e}$$

we have for the J_2 perturbation

$$\tilde{R} = \tilde{n}^2J_2R^2\big/4(1-\tilde{e}^2)^{-\frac{3}{2}}\left(2-3s_{\tilde{i}}^2\right) \tag{104}$$

Therefore,

$$\frac{\partial\tilde{R}}{\partial\tilde{a}} = -\frac{3}{4}\frac{\tilde{n}^2}{\tilde{a}}J_2R^2(1-\tilde{e}^2)^{-\frac{3}{2}}\left(2-3s_{\tilde{i}}^2\right) \tag{105}$$

$$\frac{\partial\tilde{R}}{\partial\tilde{e}} = \frac{3}{4}\tilde{n}^2J_2R^2(1-\tilde{e}^2)^{-\frac{5}{2}}\tilde{e}\left(2-3s_{\tilde{i}}^2\right) \tag{106}$$

which yields after some manipulations the expression for $\dot{\tilde{M}}_{0_{J_2}}$ as

$$\dot{\tilde{M}}_{0_{J_2}} = \tfrac{3}{2}J_2R^2\tilde{p}'^{-2}\tilde{n}\left[1-\tfrac{3}{2}s_{\tilde{i}}^2\right](1-\tilde{e}^2)^{\frac{1}{2}} \tag{107}$$

which is identical to Eq. (94). Similarly, from

$$\dot{\tilde{\lambda}}_{0_{J_2}} = -\frac{2}{\tilde{n}\tilde{a}}\frac{\partial\tilde{R}}{\partial\tilde{a}} + \frac{(1-\tilde{e}^2)^{\frac{1}{2}}\left[1-(1-\tilde{e}^2)^{\frac{1}{2}}\right]}{\tilde{n}\tilde{a}^2\tilde{e}}\frac{\partial\tilde{R}}{\partial\tilde{e}} + \frac{\tan(\tilde{i}/2)}{\tilde{n}\tilde{a}^2(1-\tilde{e}^2)^{\frac{1}{2}}}\frac{\partial\tilde{R}}{\partial\tilde{i}} \tag{108}$$

and using

$$\frac{\partial\tilde{R}}{\partial\tilde{i}} = -\frac{3\tilde{n}^2J_2R^2}{2}(1-\tilde{e}^2)^{-\frac{3}{2}}s_{\tilde{i}}c_{\tilde{i}} \tag{109}$$

the expression for $\dot{\tilde{\lambda}}_{0_{J_2}}$ is obtained after some manipulations

$$\dot{\tilde{\lambda}}_{0_{J_2}} = \tfrac{3}{4}(\tilde{n}/\tilde{a}^2)J_2R^2(1-\tilde{e}^2)^{-\frac{3}{2}}\left(2-3s_{\tilde{i}}^2\right) + \tfrac{3}{2}(\tilde{n}/\tilde{a}^2)J_2R^2(1-\tilde{e}^2)^{-2}\left[2-c_{\tilde{i}}-\tfrac{5}{2}s_{\tilde{i}}^2\right] \tag{110}$$

This expression is equivalent to $\dot{\tilde{\lambda}}_{J_2} = \dot{\tilde{M}}_{J_2} + \dot{\tilde{\omega}}_{J_2} + \dot{\tilde{\Omega}}_{J_2}$, which is obtained by adding the quantities in Eqs. (92–94), thereby showing that $\dot{\tilde{\lambda}}_{0_{J_2}} = \dot{\tilde{\lambda}}_{J_2}$, too. We now use $\dot{L}$ in Eq. (18) of Ref. 3, which is also given by Eq. (61) in terms of equinoctial elements in the present paper. With f_h given by Eq. (3), we have with $h' = \mu^{1/2}a^{1/2}(1-e^2)^{1/2}$, $r = a(1-e^2)(1+ec_{\theta^*})^{-1}$, and $\theta = \omega + \theta^*$. Also,

$$\dot{L} = (h'/r^2) + (r/h')s_\theta\tan(i/2)f_h$$

$$\dot{L} = n(1-e^2)^{-\frac{3}{2}}(1+ec_{\theta^*})^2 - \left(3nJ_2R^2\big/a^2\right) \times (1-e^2)^{-\frac{7}{2}}\tan(i/2)s_ic_is_{\omega+\theta^*}^2(1+ec_{\theta^*})^3 \tag{111}$$

such that

$$\dot{\tilde{L}} = n(1-e^2)^{-\frac{3}{2}}\frac{1}{2\pi}\int_0^{2\pi}(1+ec_{\theta^*})^2\,\mathrm{d}\theta^* - \frac{3nJ_2R^2}{a^2}(1-e^2)^{-\frac{7}{2}} \times\tan\left(\frac{i}{2}\right)s_ic_i\frac{1}{2\pi}\int_0^{2\pi}s_{\omega+\theta^*}^2(1+ec_{\theta^*})^3\,\mathrm{d}\theta^*$$

The integrations yield

$$\frac{1}{2\pi}\int_0^{2\pi}(1+ec_{\theta^*})^2\,\mathrm{d}\theta^* = \frac{1}{2}(2+e^2)$$

$$\frac{1}{2\pi}\int_0^{2\pi}s_{\omega+\theta^*}^2(1+ec_{\theta^*})^3\,\mathrm{d}\theta^* = \frac{1}{2}+\frac{3}{8}e^2+\frac{3}{4}e^2s_w^2$$

which are replaced in $\dot{\tilde{L}}$ to yield

$$\dot{\tilde{L}} = n(1-e^2)^{-\frac{3}{2}}[1+(e^2/2)] - \left(3nJ_2R^2\big/a^2\right) \times(1-e^2)^{-\frac{7}{2}}(1-c_i)c_i\left(\tfrac{1}{2}+\tfrac{3}{8}e^2+\tfrac{3}{4}e^2s_\omega^2\right) \tag{112}$$

This expression can also be converted to a form that involves the equinoctial elements themselves instead of the classical elements.

However, we need an expression for s_ω in terms of h, k, p, and q. From the definition of h and k, we have

$$s_{\omega+\Omega} = h/(h^2+k^2)^{\frac{1}{2}} = h/e, \qquad c_{\omega+\Omega} = k/(h^2+k^2)^{\frac{1}{2}} = k/e$$

which yield

$$s_w = (h/e)c_\Omega - (k/e)s_\Omega, \qquad c_w = (h/e)s_\Omega + (k/e)c_\Omega$$

and because

$$s_\Omega = p/(p^2+q^2)^{\frac{1}{2}}, \qquad c_\Omega = q/(p^2+q^2)^{\frac{1}{2}}$$

they can be written as

$$s_\omega = \frac{hq - kp}{(h^2+k^2)^{\frac{1}{2}}(p^2+q^2)^{\frac{1}{2}}} \tag{113}$$

$$c_\omega = \frac{hp + kq}{(h^2+k^2)^{\frac{1}{2}}(p^2+q^2)^{\frac{1}{2}}} \tag{114}$$

If we use $c_i = (1-p^2-q^2)(1+p^2+q^2)^{-1}$ and the s_ω expression in Eq. (113), then the $\dot{\tilde{L}}$ [Eq. (112)] will be converted to the form

$$\dot{\tilde{L}} = n(1-e^2)^{-\frac{3}{2}}\left(1+\frac{e^2}{2}\right) - \frac{3n(1-e^2)^{-\frac{7}{2}}}{a^2}\frac{J_2R^2(1-p^2-q^2)}{(1+p^2+q^2)}$$
$$\times\left(p^2+q^2+\frac{9}{4}(h^2q^2+k^2p^2)+\frac{3}{4}(k^2q^2+h^2p^2)-3hkpq\right) \tag{115}$$

As a further verification, we can start from $\dot{L}$ in Eq. (61) and f_h in Eq. (6), which are directly written in terms of h, k, p, q, and L, and using $r = a(1-h^2-k^2)(1+hs_L+kc_L)^{-1}$ write

$$\dot{L} = \frac{n(1+hs_L+kc_L)^2}{(1-e^2)^{\frac{3}{2}}} - \frac{6n(1-e^2)^{-\frac{7}{2}}J_2R^2(1-p^2-q^2)}{a^2(1+p^2+q^2)^2}$$
$$\times(qs_L - pc_L)^2(1+hs_L+kc_L)^3$$

The averaging procedure requires the following integrations:

$$\frac{1}{2\pi}\int_0^{2\pi}(1+hs_L+kc_L)^2\,\mathrm{d}L = 1+\left(\frac{h^2+k^2}{2}\right) = 1+\frac{e^2}{2}$$

$$\frac{1}{2\pi}\int_0^{2\pi}(qs_L-pc_L)^2(1+hs_L+kc_L)^3\,\mathrm{d}L$$
$$= \left(\frac{p^2+q^2}{2}\right)+\frac{9}{8}(h^2q^2+k^2p^2)$$
$$+\frac{3}{8}(k^2q^2+h^2p^2)-\frac{3}{2}hkpq$$

providing the averaged rate for L as in Eq. (115).

Therefore,

$$\dot{\tilde{L}} = n(1-e^2)^{-\frac{3}{2}}[1+(e^2/2)] + \dot{\tilde{L}}_{J_2} \tag{116}$$

with

$$\dot{\tilde{L}}_{J_2} = -\frac{3\tilde{n}\tilde{a}^{-2}J_2R^2(1-\tilde{p}^2-\tilde{q}^2)}{(1-\tilde{h}^2-\tilde{k}^2)^{\frac{7}{2}}(1+\tilde{p}^2+\tilde{q}^2)^2}\Bigg(\tilde{p}^2+\tilde{q}^2$$
$$+\frac{9}{4}(\tilde{h}^2\tilde{q}^2+\tilde{k}^2\tilde{p}^2)+\frac{3}{4}(\tilde{k}^2\tilde{q}^2+\tilde{h}^2\tilde{p}^2)-3\tilde{h}\tilde{k}\tilde{p}\tilde{q}\Bigg) \tag{117}$$

The partial derivatives of $\dot{\tilde{L}}_{J_2}$ are now readily obtained as

$$\frac{\partial\dot{\tilde{L}}_{J_2}}{\partial\tilde{a}} = -\frac{7}{2\tilde{a}}\dot{\tilde{L}}_{J_2} \tag{118}$$

$$\frac{\partial\dot{\tilde{L}}_{J_2}}{\partial\tilde{h}} = \frac{7\tilde{h}}{(1-\tilde{h}^2-\tilde{k}^2)}\dot{\tilde{L}}_{J_2} - \frac{3\tilde{n}\tilde{a}^{-2}J_2R^2(1-\tilde{p}^2-\tilde{q}^2)}{(1-\tilde{h}^2-\tilde{k}^2)^{\frac{7}{2}}(1+\tilde{p}^2+\tilde{q}^2)^2}$$
$$\times\left(\frac{9}{2}\tilde{h}\tilde{q}^2+\frac{3}{2}\tilde{h}\tilde{p}^2-3\tilde{k}\tilde{p}\tilde{q}\right) \tag{119}$$

$$\frac{\partial\dot{\tilde{L}}_{J_2}}{\partial\tilde{k}} = \frac{7\tilde{k}}{(1-\tilde{h}^2-\tilde{k}^2)}\dot{\tilde{L}}_{J_2} - \frac{3\tilde{n}\tilde{a}^{-2}J_2R^2(1-\tilde{p}^2-\tilde{q}^2)}{(1-\tilde{h}^2-\tilde{k}^2)^{\frac{7}{2}}(1+\tilde{p}^2+\tilde{q}^2)^2}$$
$$\times\left(\frac{9}{2}\tilde{k}\tilde{p}^2+\frac{3}{2}\tilde{k}\tilde{q}^2-3\tilde{k}\tilde{p}\tilde{q}\right) \tag{120}$$

$$\frac{\partial\dot{\tilde{L}}_{J_2}}{\partial\tilde{p}} = -\frac{2\tilde{p}}{(1-\tilde{p}^2-\tilde{q}^2)}\dot{\tilde{L}}_{J_2} - \frac{4\tilde{p}}{(1+\tilde{p}^2+\tilde{q}^2)}\dot{\tilde{L}}_{J_2}$$
$$-\frac{3\tilde{n}\tilde{a}^{-2}J_2R^2(1-\tilde{p}^2-\tilde{q}^2)}{(1-\tilde{h}^2-\tilde{k}^2)^{\frac{7}{2}}(1+\tilde{p}^2+\tilde{q}^2)^2}$$
$$\times\left(2\tilde{p}+\frac{9}{2}\tilde{p}\tilde{k}^2+\frac{3}{2}\tilde{p}\tilde{h}^2-3\tilde{h}\tilde{k}\tilde{q}\right) \tag{121}$$

$$\frac{\partial\dot{\tilde{L}}_{J_2}}{\partial\tilde{q}} = -\frac{2\tilde{q}}{(1-\tilde{p}^2-\tilde{q}^2)}\dot{\tilde{L}}_{J_2} - \frac{4\tilde{q}}{(1+\tilde{p}^2+\tilde{q}^2)}\dot{\tilde{L}}_{J_2}$$
$$-\frac{3\tilde{n}\tilde{a}^{-2}J_2R^2(1-\tilde{p}^2-\tilde{q}^2)}{(1-\tilde{h}^2-\tilde{k}^2)^{\frac{7}{2}}(1+\tilde{p}^2+\tilde{q}^2)^2}$$
$$\times\left(2\tilde{q}+\frac{9}{2}\tilde{h}^2\tilde{q}+\frac{3}{2}\tilde{k}^2\tilde{q}-3\tilde{h}\tilde{k}\tilde{p}\right) \tag{122}$$

Conclusions

Precision integrated minimum-time orbit transfers between general elliptic orbits using continuous constant acceleration and subjected to the perturbing effects of the second zonal harmonic J_2 are generated with two different formulations using two different sets of equinoctial elements. The use of the true longitude as the sixth element, as well as the variable with respect to which all of the dynamic and adjoint differential equations are expressed, coupled with the use of the polar coordinate frame as the orbital frame for component resolution of the thrust and J_2-induced perturbations result in an analysis that is substantially simpler than previously developed versions and lead to simulation software that exhibits better convergence characteristics. The full six-state analysis is used to generate a free–free minimum-time transfer that is verified and compared with the results generated by a previously derived formulation. The partial derivatives of the averaged rate of the true longitude due to J_2, with respect to the five slowly varying elements, are also derived analytically for use in software that uses averaged dynamics for approximate and rapid generation of minimum-time transfers.

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